to the direction $\mathbf{P}_\gamma\times\mathbf{P}_K$. A graphical comparison of $d\sigma/d\Omega$ with the data is shown in Fig. 1(a)–(e). At $\theta=80°$, and $k_L=1.054$ BeV we find the Λ^0 20% polarized in the $\mathbf{P}_\gamma\times\mathbf{P}_K$ direction, which contradicts the measured value of 0.37 ± 0.17 in the $\mathbf{P}_K\times\mathbf{P}_\gamma$ direction[10] ($|\alpha|P=0.23\pm0.08$, and $|\alpha|=0.62\pm0.07$).[11]

SUMMARY AND CONCLUSIONS

We have introduced a model for low-energy photoproduction of Λ particles off protons. We retained only the contributions from the 1-N, 1-K, 1-K^*, and $p_{1/2}$ nucleon resonance intermediate states. The result given by (4a) for $g_{N\Lambda K}^2/4\pi=5.8$ is in acceptable agreement with Kanazawa's[8] value of 5.0. Reasonable agreement with the experimental data for the differential cross sections is obtained at most energies. However, we note that our fits to the differential cross section have one defect: It would appear that a smaller a_1 at the lower energies and slightly higher a_1 at energies above 1020 MeV would improve the fit. The model cannot supply this need while keeping the quantities a_0 and a_2 reasonable.

The model has been used to predict differential cross sections at three energies for which no data are yet available. It has also been used to predict Λ-polarization curves at three different energies, with results that contradict the datum. If this measured value of the polarization is confirmed, then it would be necessary to forsake this model.

It should be remembered that the hyperon poles and resonances and the various nucleon resonances have not been included in this study. In particular, the $f_{5/2}$ nucleon resonance at 1690 MeV, even with its large centrifugal barrier, may be instrumental in producing large Λ polarizations. We are, therefore, continuing our investigations to assess the effects of such contributions on photoproduction and related processes.

We are most grateful to J. E. Rush for checking some of the numerical calculations.

[11] J. W. Cronin and O. W. Overseth, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 453.

PHYSICAL REVIEW VOLUME 131, NUMBER 6 15 SEPTEMBER 1963

Gauge Invariance and Integration Rules*

F. C. KHANNA AND F. ROHRLICH†
Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa

(Received 6 May 1963)

The Feynman-Dyson rules of integration for the scattering matrix elements in perturbation expansion have long been known to lead to gauge-*variant* results in the case of certain closed loop diagrams. It is shown that a more careful integration which avoids unjustified interchanges of integrations and limiting processes keeps the theory gauge-invariant throughout, without the need of explicit cutoffs or appeal to invariance for the specification of undefined integrals. The Feynman-Dyson rules can thus easily be amended to assure gauge invariance.

THE work of Tomonaga, Schwinger, and Feynman about 15 years ago led to a milestone in the development of quantum field theory: It was finally possible to predict observable effects due to radiative corrections. The success of their work must be attributed, to a large extent, to the extensive use of the invariance properties of the theory. For this reason, it seems contradictory to observe that this same theory is unable to provide gauge-invariant results without explicit help by the "better-knowing" theoretician. The present paper is intended to remedy this situation.

The difficulty appears in those calculations which involve divergent closed loop diagrams. Specifically, the closed loops with two corners lead to a nonvanishing photon self-energy and the one with four corners provides terms to the photon-photon scattering cross section which depends on the potentials rather than the fields. While a gauge-independent quantum electrodynamics, based on field strengths,[1] can be formulated in a covariant manner[2] and would, therefore, avoid this difficulty, there is no reason why the usual Schwinger-Feynman-Dyson formulation should not carry through in a gauge-invariant way. Although the fundamental equations are gauge invariant, the usual integrations result in gauge-dependent terms, so that this invariance property must have been lost in the integration process.

The usual attitude is to consider oneself helpless in view of the appearing divergences, which can easily be blamed for this difficulty as they have been blamed for

* This work was supported in part by a grant from the National Science Foundation.

† Present address: Department of Physics, Syracuse University Syracuse, New York.

[1] F. J. Belinfante and J. S. Lomont, Phys. Rev. **84**, 541 (1951); F. J. Belinfante, *ibid.* **84**, 546 (1951).

[2] F. Rohrlich, State University of Iowa Res. Rept. 62–15 (unpublished); in Proceedings of the Midwest Theory Conference, Argonne, 1962 (unpublished).

all other difficulties in the past. However, the success of the theory on this level of development consists in a *set of rules* which specify how to integrate and how to renormalize, so as to obtain unambiguous finite results. While this is clearly an unsatisfactory state of the theory, it would be inadmissible if these rules were faulty. This is the motivation of the following remarks.

We resolve the difficulty by observing that closed loops with $n>4$ corners yield convergent integrals, but would still result in gauge-dependent results if the limit $k\rightarrow 0$ were interchanged with the integrations, resulting in the confluence of singularities. What is done in the following pages consists simply in a formal extension of the evaluation procedure for convergent loops to the divergent integrals of loops of two and four corners.

THE TWO-CORNER LOOP

The scattering matrix in perturbation expansion can be integrated by use of the Feynman-Dyson rules. These differ from *ad hoc* rules used in specific calculations in that they are valid to all orders of the expansion.

Using these rules the gauge-variant photon self-energy arises in second order.[3] Specifically, the two-cornered loop yields the symmetric tensor $\tilde{\Pi}_{\mu\nu}(k)$ in momentum space which must be Lorentz invariant as well as gauge invariant. These requirements imply, according to well-known arguments,[3] that $\tilde{\Pi}_{\mu\nu}(k)$ has the form

$$\tilde{\Pi}_{\mu\nu}(k)=(k_\mu k_\nu-k^2g_{\mu\nu})C(k) \tag{1}$$

and that

$$\lim_{k\rightarrow 0}\tilde{\Pi}_\mu{}^\mu(k)=0. \tag{2}$$

The diffculty now arises from the integration of

$$\tilde{\Pi}_\mu{}^\mu(k)=-\frac{2\alpha}{i\pi^3}\int\frac{p^2+2m^2-p\cdot k}{[(p-k)^2+m^2][p^2+m^2]}d^4p\,, \tag{3}$$

which, apparently, does not lead to the result (2), in violation of gauge invariance.

However, the usual deduction of $\tilde{\Pi}_{\mu\nu}(0)\neq 0$ from (3) is invalid for the following reason. This deduction, as usually carried out, involves an unjustified interchange of the limit $k\rightarrow 0$ with the p integration. If this interchange is carried out a confluence of the poles of the integrand results and the integral diverges. On the other hand, a more careful evaluation results in the gauge-invariant equation (2). This can be seen as follows:

As is well known, the Feynman contour is equivalent to the replacement

$$\frac{1}{p^2+m^2-i\epsilon}\longrightarrow P\frac{1}{p^2+m^2}+i\pi\delta(p^2+m^2)\,,$$

with subsequent integration along the real p^0 axis. P indicates the Cauchy principal value. The integral (3) then separates into a real and an imaginary part,

$$\mathrm{Re}\tilde{\Pi}_\mu{}^\mu(k)=-\frac{2\alpha}{\pi^2}P\int d^4pd^4p'\,\delta(p-p'-k)(p\cdot p'+2m^2)\times\left[\frac{\delta(p^2+m^2)}{p'^2+m^2}+\frac{\delta(p'^2+m^2)}{p^2+m^2}\right], \tag{4}$$

$$\mathrm{Im}\tilde{\Pi}_\mu{}^\mu(k)=+\frac{2\alpha}{\pi^3}\int d^4pd^4p'\,\delta(p-p'-k)(p\cdot p'+2m^2)\times\left[P\frac{1}{p^2+m^2}P\frac{1}{p'^2+m^2}-\pi^2\delta(p^2+m^2)\delta(p'^2+m^2)\right]. \tag{5}$$

This imaginary part is known[4] to vanish for $k=0$. In fact, it vanishes whenever $k^2>-4m^2$.

However, the real part also vanishes when one observes that

$$P\int f(x)\delta(x)dx=0\,,\quad\text{or symbolically}\quad P\delta(x)=0\,, \tag{6}$$

for all functions $f(x)$ which have at most a simple pole at $x=0$. The Cauchy principal part requires an integral over an interval which excludes exactly the nonvanishing contributions of the δ function. Thus, the limit $k\rightarrow 0$ can here even be interchanged with the integration, resulting in the well defined

$$\mathrm{Re}\tilde{\Pi}_\mu{}^\mu(0)=-\frac{4\alpha}{\pi^2}P\int d^4p(p^2+2m^2)\frac{\delta(p^2+m^2)}{p^2+m^2}=0. \tag{7}$$

The photon self-energy thus vanishes and the gauge invariance requirement (2) is satisfied.

THE FOUR-CORNER LOOP

The four-corner loop involves the sum of three diagrams.[5] The corresponding polarization tensor $T_{\mu\nu\lambda\sigma}(k_1k_2k_3k_4)$ must be Lorentz invariant and gauge invariant. The latter condition requires that the expansion in powers of small k,

$$T_{\mu\nu\lambda\sigma}=T_{\mu\nu\lambda\sigma}(0)+k^\alpha k^\beta\frac{\partial^2}{\partial k^\alpha\partial k^\beta}T_{\mu\nu\lambda\sigma}\bigg|_{k=0}+k^\alpha k^\beta k^\gamma k^\delta\frac{\partial^4T_{\mu\nu\lambda\sigma}}{\partial k^\alpha\partial k^\beta\partial k^\gamma\partial k^\delta}\bigg|_{k=0}+\cdots, \tag{8}$$

or symbolically,[6]

$$T=T(0)+kkT''(0)+kkkkT''''(0)+\cdots,$$

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[3] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), especially Sec. 9–5.

[4] G. Källen, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 6, Part 1, Sec. 29.

[5] J. M. Jauch and F. Rohrlich, Ref. 3, Sec. 13–1.

[6] The odd powers in this expansion are excluded by Lorentz invariance.

cannot contain terms of lower order than the fourth. Thus, $T(0)$ and $T''(0)$ in (8) must vanish identically. This can be proven to be the case[5] for the divergent quantity $T(0)$, provided one sums over all the diagrams. But for $T''(0)$ this cannot be proven without appeal to a gauge-invariant cutoff.

The expression for $T_{\mu\nu\lambda\sigma}$ is well known and can be written as[7]

$$T_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4) = \frac{-2e^4}{(2\pi)^6}\int d^4p\, \mathrm{Tr}[\gamma_\mu S_{1R}(p)\gamma_\nu S_{1R}(p-k_2) \times\gamma_\lambda S_{1R}(p-k_2-k_3)\gamma_\sigma S_{1R}(p+k_1)],$$

where

$$S_{1R}(p)=1/(i\mathbf{p}+m)$$

and the integration over p is to be carried out along the Feynman contour. If, once again, we make the replacement

$$S_{1R}(p)\rightarrow S_P(p)+\tfrac{1}{2}iS_1(p) \tag{9}$$

we find that in the limit k_1, k_2, k_3, $k_4\rightarrow 0$ $\mathrm{Im}T_{\mu\nu\lambda\sigma}\times(k_1k_2k_3k_4)$ vanishes, but that the real part contains integrals of the type

$$\int d^4p\, \mathrm{Tr}[\gamma_\mu S_P\gamma_\nu S_P\gamma_\lambda S_P\gamma_\sigma S_P] \tag{10a}$$

and

$$\int d^4p\, \mathrm{Tr}[\gamma_\mu S_1\gamma_\nu S_1\gamma_\lambda S_1\gamma_\sigma S_1]. \tag{10b}$$

Integrals containing both S_P and S_1 will vanish because of (6). The expressions (10) become infinite if we interchange the integration and the limit k_1, k_2, k_3, $k_4\rightarrow 0$. But they vanish when such an interchange is not made, as is the case in (5). This allows us to write

$$\lim_{k_1,k_2,k_3,k_4\rightarrow 0} T_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4)=0. \tag{11}$$

This proves that $T(0)=0$ for each diagram and not only for their sum as in Ref. 6.

In order to prove that the second term on the right side of (8) vanishes, we make use of the identity

$$(\partial/\partial p_\mu)S(p)=iS(p)\gamma^\mu S(p). \tag{12}$$

For example,

$$\partial_{k_2}{}^\beta\partial_{k_1}{}^\alpha T_{\mu\nu\lambda\sigma}=\frac{-2e^4}{(2\pi)^6}\int d^4p\, \mathrm{Tr}\{\gamma_\mu S_{1R}(p)\gamma_\nu S_{1R}(p-k_2)\gamma^\beta S_{1R}(p-k_2)\gamma_\lambda S_{1R}(p-k_2-k_3)\gamma_\sigma S_{1R}(p+k_1)\gamma^\alpha S_{1R}(p+k_1) + \gamma_\mu S_{1R}(p)\gamma_\nu S_{1R}(p-k_2)\gamma_\lambda S_{1R}(p-k_2-k_3)\gamma^\beta S_{1R}(p-k_2-k_3)\gamma_\sigma S_{1R}(p+k_1)\gamma^\alpha S_{1R}(p+k_1)\}.$$

The limit $k\rightarrow 0$ of this expression now poses exactly the same problem as the *leading* term of the six-corner loop and can, therefore, be evaluated exactly like $T(0)$. In this way we find that

$$\lim_{k_1,k_2,k_3,k_4\rightarrow 0} \partial_{k_2}{}^\beta\partial_{k_1}{}^\alpha T_{\mu\nu\lambda\sigma}(k_1,k_2,k_3,k_4)=0. \tag{13}$$

Therefore, the expression $T_{\mu\nu\lambda\sigma}$ for the four-corner loop can be expressed in a Lorentz invariant and gauge invariant manner, provided the integrals are evaluated without the unjustified interchange of the limit $k\rightarrow 0$ with the integrations. We want to stress here the important role played by the identity (12) in proving gauge invariance of four-cornered loops; it is closely related to Ward's identity.

CONCLUSION

It is obvious that the above method applies equally to loops with $n>4$ corners. While such loops do not yield divergent integrals, they do yield gauge-dependent terms unless the integrations are carried out carefully as indicated above.

The usual Feynman-Dyson rules do not specify methods to handle the product of two or more than two S_{1R} functions of the same argument. In momentum space, each $\tilde{S}_{1R}(p)$ function has to be integrated along a Feynman contour. Confluence of the singularities of two different S_{1R} functions before all the integrations are carried out leads to gauge-dependent (infinite) terms in the S matrix. Therefore, we must specify the rules in such cases:

(a) Confluence of singularities of different S_{1R} functions is not permissible until all the integrations have been carried out; the interchange of this limiting process with the integrations is not correct.

(b) Principal value integrals involving distributions must be carried out consistently according to Eq. (6).

[7] This is the same expression as Eq. (13.4) of Ref. 3.